Prismatic structural members: Cross section flat and normal to the axis?

Piezas prismáticas estructurales: ¿Sección transversal plana y normal al eje?

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Abstract
In the study of prismatic bars subjected to bending forces, the authors of *Strength of Materials* generally assume the Navier-Bernoulli simplifying hypothesis which states that flat cross sections (CS) normal to their axes before deformation remain flat and normal to their axes. A more detailed study in terms of *Elasticity*, however, shows how approximate this hypothesis can be for some basic prismatic bar problems in which displacements can readily be obtained. When or whether the surface remains flat, absolutely flat, or not is a point of debate among engineers and architects alike and even for structural specialists, who look deeper into this kind of issues. This paper proposes a detailed study of said problems and clarifies them. Contrary to what should be expected according to well-established literature, the CS of any bar subjected to pure bending forces does not remain flat after deformation. Our analysis revisits accepted displacement solutions for tension, bending and torque and will hopefully remove the misunderstanding leading to a flat geometry for the deformed CS. It also includes the correct interpretation of the warped geometry from the exact equations we obtain in this paper, which we illustrate with results from finite elastic models.

Key words*:* Prismatic member, bending, flat cross section, warping.

Resumen
En el estudio de las barras sometidas a flexión, la *Resistencia de Materiales* asume la hipótesis simplificadora de Navier-Bernoulli. Esta indica que secciones planas transversales (ST) al eje de la pieza antes de la deformación permanecen planas y normales al eje deformado tras ella. Un análisis más detallado en términos de la *Elasticidad* muestra el carácter aproximado de esta hipótesis para algunos problemas elementales de barras, en los cuales los movimientos se pueden obtener fácilmente. Esta discusión es conocida por arquitectos e ingenieros, pero en muchos casos no está claro, incluso para especialistas estructurales, en qué casos la ST permanece o no "exactamente" plana. El artículo discute los problemas citados y aclara la situación. Como conclusión inesperada se obtiene que, incluso en flexión pura, la ST no permanece plana tras deformarse, como se supone en tratados muy clásicos. El análisis se realiza a partir de soluciones de movimientos conocidas para tracción, flexión y torsión. La discusión aclara el malentendido que conduce a una geometría plana para la ST deformada e incluye la interpretación correcta de su geometría alabeada, a partir de las ecuaciones exactas obtenidas en este artículo e ilustradas con resultados de modelos de elementos finitos elásticos.

Palabras clave: Pieza prismática, flexión, sección transversal plana, alabeo.

Introduction

The most important hypothesis assumed in *Strength of Materials* treatises in order to the simplified mathematical model for slender bar bending states that flat cross sections normal to the axis before loading remain flat and normal to the deformed axis after loading.

Timoshenko (1940) explains this assumption from the experimental measurements of deformed geometry of straight lines plotted on the lateral bar prior to pure bending (with no shear force involved). Good accordance of these subtle displacements with a linear behaviour leads to a plausible model which is able to evaluate strains and stresses with significant precision. However, the fact is that in many cases the cross section (CS) does not remain 100% flat and that warping occurs though not easily perceivable.

Feodosiev (1980) and Argüelles & Viña (2004) explain the deformation of cross sections taking into consideration the constant value for bending moment and symmetry. However, symmetry only ensures null warping at the centre of the bar CS, but warping is free to occur at the bar ends, except when specially imposed. Then, warping may occur in every other CS even if the bending moment and stresses remain effectively constant.

On the other hand, some texts on *Elasticity* (Timoshenko & Goodier, 1951; Torroja, 1967; Samartin, 1990) present a more detailed analysis of the problem from a tri-dimensional mathematical model. For pure bending cases, equations for CS point displacements lead to a linear axial component on the transversal coordinate in the bending plane. This is considered a proof of the simplification assumed in the model proposed in *Strength of Materials*. However, it seems these studies may have overlooked the fact that the other displacement components, also calculated in *Elasticity*, displace the points of coordinates *(x, y, z)* on a CS before bending to locations *(x′, y′, z′)* on a warped surface after bending, even in pure bending.

This paper presents even a more detailed analysis of the problem. We obtain the equation for the warped surface f (x′, y′, z′) = 0, and we describe the problem. Our analysis further examines other load cases of prismatic member as axial forces (which we have found needing additional remarks in well-established literature), shear bending and torsion. Our purpose is to thoroughly discuss whether warping occurs or not in the CS using well-known elastic solutions.

Methodology

Our discussion will start with exact equations from the Elasticity Theory, with the formulation corresponding to small displacements, rotations and strains from which we obtain explicit expressions; the formulation corresponding to big values of these variables will necessarily require numerical treatment to achieve similar conclusions.

We will use well-known solutions for displacements and stresses for prismatic members:

$$\vec{c} = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix} \qquad \boldsymbol{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ SIM & & \sigma_z \end{bmatrix}_{(x,y,z)}$$

They allow writing the coordinates of material points after deformation occurs *(x′, y′, z′)* in terms of the coordinates before deformation *(x, y, z)* in a Lagrangian formulation:

$$x' = x + u \qquad (1.a)$$

$$y' = y + v \qquad (1.b)$$

$$z' = z + w \qquad (1.c)$$

The shape of the deformed CS is defined by the relationship between the coordinates after deformation

$$f\,(x', y', z') = 0 \qquad (2)$$

obtained by specifically setting $x=x_0$ (an arbitrary CS) and eliminating *y, z* from (1). The expression of displacements $\vec{c}(x,y,z)$ in terms of the old coordinates is not useful by itself when the CS surface after deformation (2) needs to be defined. This is, in our understanding, the origin of the usual misunderstanding that concludes that in several cases the CS remains flat after deformation.

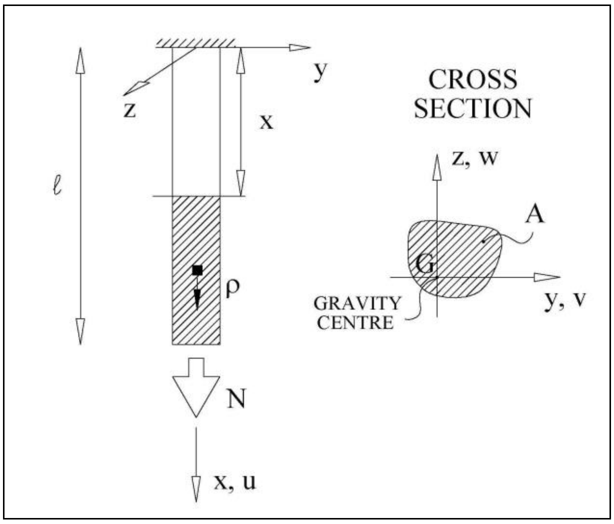
In the next section, we will discuss the cases of a straight bar subjected to pure tension, pure bending, shear bending and torsion --with special emphasis on pure tension and pure bending-- to obtain the corresponding analytical expressions (2).

Finally, we will present the results of a finite element numerical analysis for those cases presented to illustrate the CS predicted by the exact expressions. Since the displacements are barely noticeable, the graph scale is distorted to be able to appreciate the curvature of the surface corresponding to the deformed CS.

Axial load

To include cases of uniform and variable axial force, let us consider the bar subjected to the action of forces ***N*** (at the gravity centre at the end of CS) and its own weight **ρ** as seen in Figure 1.

Figure 1. Prismatic member under variable axial force.
Source: Self-Elaboration.



For the purpose of explanation, both tension and compression differ from each other only in the sign for stresses and displacements. The elemental elastic solution for uniform CS is available in literature for stresses ***T***, strains ***D*** and displacements *(u, v, w),* (Timoshenko & Goodier, 1951; Torroja, 1967) and it is expressed:

$$\boldsymbol{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ SIM & & \sigma_z \end{bmatrix} = \begin{bmatrix} \dfrac{N+\rho(l-x)}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ & \varepsilon_y & \gamma_{yz}/2 \\ SIM & & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \sigma_x/E & 0 & 0 \\ 0 & -\upsilon\sigma_x/E & 0 \\ 0 & 0 & -\upsilon\sigma_x/E \end{bmatrix}$$

(E = elastic modulus, υ =Poisson ratio)

$$u = \int \varepsilon_x dx = \frac{1}{EA}\left[(N+\rho\, l)x - \frac{\rho\, x^2}{2} - \frac{\upsilon\rho}{2}(y^2+z^2)\right] \tag{3.a}$$

$$v = \int \varepsilon_y dy = -\frac{\upsilon}{EA}[N+\rho(l-x)]y \tag{3.b}$$

$$w = \int \varepsilon_z dz = -\frac{\upsilon}{EA}[N+\rho(l-x)]z \tag{3.c}$$

Expressions (3) are obtained for the case of zero displacements and rotations at origin.

A point *P(x, y, z)* before deformation will be placed at *P' (x', y', z')* after it occurs, being:

$$x' = x + u = x + \frac{1}{EA}\left[(N+\rho\, l)x - \frac{\rho\, x^2}{2} - \frac{\upsilon\rho}{2}(y^2+z^2)\right] \tag{4.a}$$

$$y' = y + v = y - \frac{\upsilon}{EA}\left[N+\rho(l-x)\right] y \tag{4.b}$$

$$z' = z + w = z - \frac{\upsilon}{EA}\left[N+\rho(l-x)\right] z \tag{4.c}$$

Description of the deformed surface corresponding to an initially flat CS normal to the axis is not possible directly from (3.a); it requires the relationship between *x'*, *y'* and *z'*, *f (x', y', z') =0*, for the corresponding $x=x_0$ value, substituting it in (4) and eliminating *y* and *z* from (4.b) and (4.c) respectively as explained in the Methodology section:

$$y = \frac{y'}{1-\frac{\upsilon}{EA}\left[N+\rho(l-x_0)\right]}$$

$$z = \frac{z'}{1-\frac{\upsilon}{EA}\left[N+\rho(l-x_0)\right]}$$

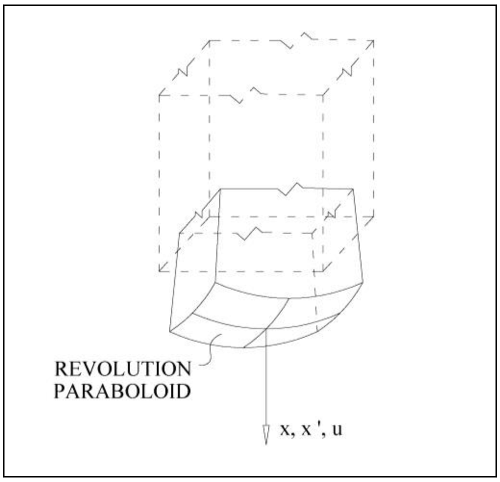
And substituting in (4.a) we obtain:

$$x' = x_0 + \frac{1}{EA}\left[(N+\rho l)x_0 - \rho x_0^2/2\right] - \frac{\rho\upsilon EA}{2}\frac{y'^2+z'^2}{\left\{EA-\upsilon\left[N+\rho(l-x_0)\right]\right\}^2} \tag{5}$$

Equation (5) reveals that the deformed CS surface is, in general case and regardless of shape, a part of a revolution paraboloid, as related in cited literature. It is shown in fig. 2 for a rectangular CS. But instead of being derived from the quadratic expression (3.a), it derives from the quadratic expression (5).

Only in the absence of its own weight, $\rho = 0$ in (5), this surface remains exactly flat and normal to the axis after deformation.

Figure 2. Paraboloidal deformed CS under linear axial force. Source: Self-Elaboration.



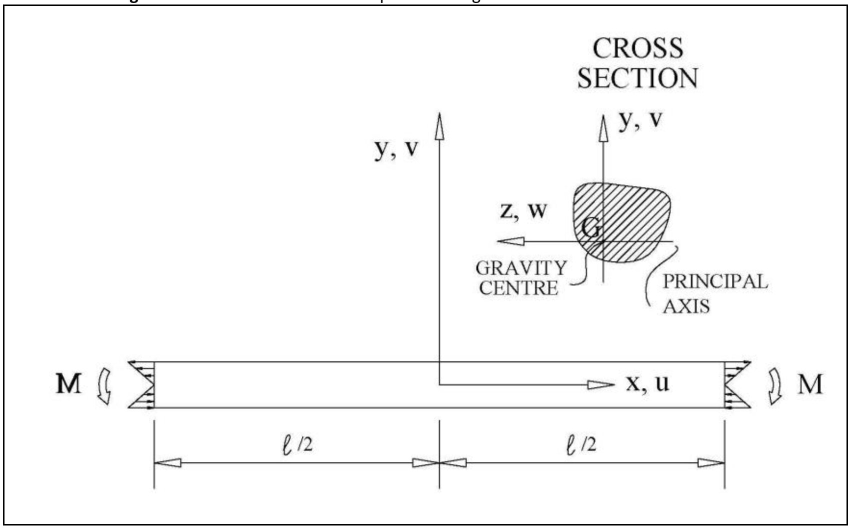
In this case, simplified calculations for axial displacement assuming that CS remains flat (although not being the case) reproduce exactly the values (3) only for the member axis (*y=z=0*); it fails for other material points on the CS, obviously. On the contrary, stresses obtained by simplified theory coincide with the exact values in the entire CS.

In the precedent discussion ***N*** was assumed to act uniformly throughout the CS area with uniform σ_x. Otherwise, the Saint Venant principle needs to be considered to assume the given solution.

Pure bending

The elastic solution for pure bending in the case of uniform CS (Figure 3) is also available in the literature (Timoshenko & Goodier, 1951; Torroja, 1967; Samartin, 1990). For bending around a central and principal axis of a CS, we have:

Figure 3. Prismatic member under pure bending. Source: Self-Elaboration.



T = $\begin{bmatrix} \frac{M}{I}y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (*I* = inertia moment)

D = $\begin{bmatrix} \sigma_x/E & 0 & 0 \\ 0 & -\upsilon\sigma_x/E & 0 \\ 0 & 0 & -\upsilon\sigma_x/E \end{bmatrix}$

$$u = \frac{M}{EI}xy = kxy \quad (6.a)$$

$$v = \upsilon k(z^2 - y^2 - x^2/\upsilon)/2 \quad (6.b)$$

$$w = -\upsilon kyz \quad (6.c)$$

with $k = \frac{M}{EI} > 0$

Again, expressions (6) are obtained for the case of zero displacements and rotations at origin. For any point in the bar *P(x, y, z)* before deformation, its deformed counterpart will be *P'(x', y', z')* with:

$$x' = x + u = x + kxy \quad (7.a)$$

$$y' = y + v = y + \upsilon k(z^2 - y^2 - x^2/\upsilon)/2 \quad (7.b)$$

$$z' = z + w = z - \upsilon kyz \quad (7.c)$$

As explained in the methodology section, the CS deformed surface will be depicted by *f (x', y', z') =0* for a given $x=x_0$. Substituting it in (7) and eliminating *y* and *z* from (7.a) and (7.c), respectively, we have:

$$y = \frac{1}{k}(\frac{x'}{x_0} - 1) \qquad (7.b')$$

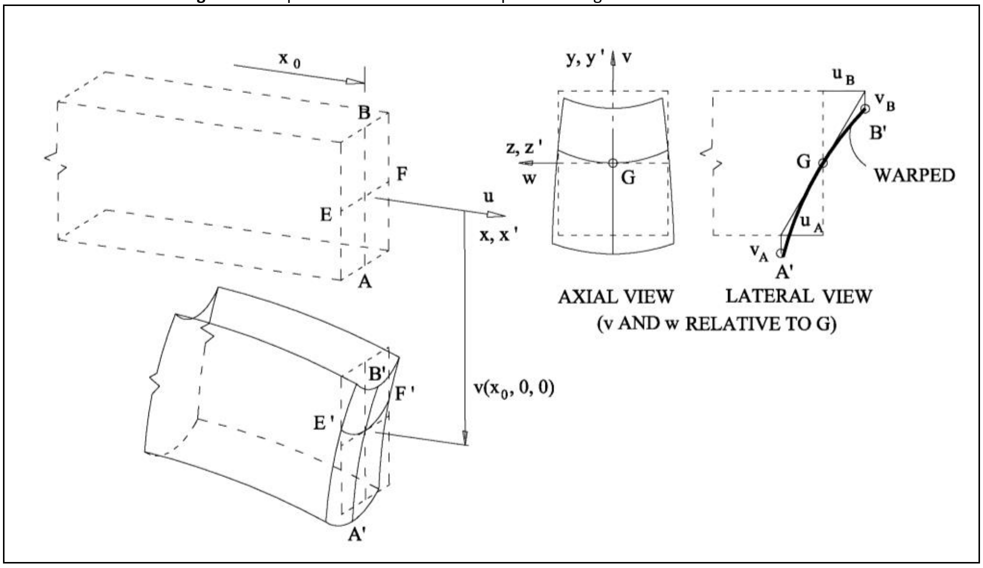
$$z = \frac{z'}{1 - \upsilon k y} = \frac{z'}{1 - \upsilon(\frac{x'}{x_0} - 1)} \qquad (7.c')$$

And substituting in (7.b) we reach:

$$z'^2 = \left[1 - \upsilon(\frac{x'}{x_0} - 1)\right]^2 \cdot \left[\frac{1}{k^2}(\frac{x'}{x_0} - 1)^2 - \frac{2}{\upsilon k^2}(\frac{x'}{x_0} - 1) + \frac{2}{\upsilon k} y' + \frac{x_0^2}{\upsilon}\right] \qquad (8)$$

Expression (8) corresponds to a polynomial surface of 4th order that contains several parabolas families, as we will show. Note that this line of discussion is valid for any CS shape. As an example, Figure 4 illustrates this fact for a rectangular CS. Effectively:

Figure 4. Warped CS after deformation in pure bending. Source: Self-Elaboration.



a) Straight lines in the CS parallel to neutral axis EF ($x = x_0$; $y = y_0$) are transformed to parabolas in different planes parallel to the undeformed CS:

   from (7.a) $x' = x_0(ky_0 + 1)$, constant (plane)

   from (8) and last equation $z'^2 = (1 - \upsilon k y_0)^2 (y_0^2 + x_0^2 / \upsilon - 2y_0 / \upsilon k + 2y' / \upsilon k)$

   (parabola with axis parallel to *y′*, see axial view in Figure 4).

b) Straight lines in the CS normal to neutral axis ($x = x_0$; $z = z_0$) are transformed to parabolas in different planes oblique to undeformed *x* axis (except the one corresponding to the line AB, which obviously corresponds to the symmetry plane):

   from (7c′) $\frac{1}{\upsilon}(1 - \frac{z'}{z_0}) = \frac{x'}{x_0} - 1$ (plane oblique to the axis)

   from (8) and last equation $z_0^2 - x_0^2 / \upsilon = \frac{1}{\upsilon^2 k^2}(1 - z'/z_0)^2 - \frac{2}{\upsilon^2 k^2}(1 - z'/z_0) + \frac{2}{\upsilon k} y'$

   (parabola with axis parallel to *y′*, see axial view in Figure 4).

For just the line *AB* in the symmetry plane, $z = z_0 = 0$ and the previous general equation is not valid. For this line, from (7.c): $z' = 0$ (symmetry plane), and the 2nd bracket in (8) must be zero:

$$y' = -\frac{k}{2}x_0^2 + \frac{1}{k}\left(\frac{x'}{x_0} - 1\right) - \frac{\upsilon}{2k}\left(\frac{x'}{x_0} - 1\right)^2 \tag{9}$$

(parabola in x-y plane, see lateral view in Figure 4).

On the other hand, as it is well known, the straight lines parallel to the axis prior to deformation ($y = y_0$, $z = z_0$) also turn after it into parabolas in planes parallel to the bending plane after deformation takes place:

from (7.c) $z' = z_0(1 - \upsilon k y_0)$, constant (plane parallel to the bending one)

from (7.a and b) $x = \dfrac{x'}{1 + k y_0}$

$y' = y_0 + \dfrac{\upsilon k}{2}\left[z_0{}^2 - y_0{}^2 - \dfrac{x'^2}{\upsilon(1 + k y_0)^2}\right]$ (parabola with axis parallel to y')

In the case of pure bending, analysis in some texts hastily conclude by stating that all CS remain flat after deformation.

In *Strength of Materials* treatises (Feodosiev, 1980; Argüelles & Viña, 2004) we find considerations derived from symmetry for any part of the beam to conclude that uniform bending and deformation occur along the entire bar and that the CS is always flat. But it is not precise in general according to our calculations: the symmetry plane in the middle of the beam imposes the null warping condition, but at the ends warping is free (except when specially imposed). In reality, warping is not uniform. It can be uniform only when warping is declared null at the ends. It is evident, nevertheless, that the static variables (stress and bending load) are effectively uniform in any cases.

In *Elasticity* treatises (Timoshenko & Goodier, 1951; Torroja, 1967; Samartin, 1990) the conclusion derives again from the interpretation that a linear expression on *y* for *u* displacement in (6.a) ($x=x_0$ being a constant in the CS) corresponds to a flat CS after deformation.

But different $v(x_0, y, z)$ and $w(x_0, y, z)$ for different CS points cause section warping (Figure 4, lateral view), *A'* and *B'* being the location after deformation of *A* and *B*, respectively: cross section is not flat any more. Correct interpretation should be made applying (8).

Finally, from equation (9) we can appreciate the relative importance of warping equation terms by calculating the curvature of the deformed CS:

$$k_{CS} = \frac{d^2y'/dx'^2}{[1 + (dy'/dx')^2]^{3/2}} = \frac{-\upsilon k^2 x_0}{[k^2x_0^2 + 1 + \upsilon^2(x'/x_0 - 1)^2 - 2\upsilon(x'/x_0 - 1)]^{3/2}} \approx -\upsilon k^2 x_0 \tag{10}$$

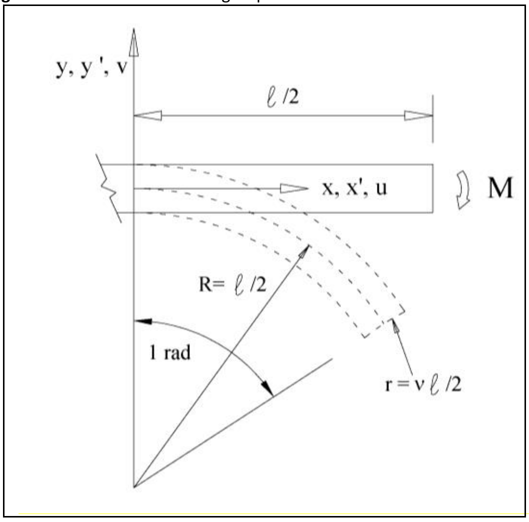
where small terms in the denominator are neglected ($kx_0 << 1$, $(x'/x_0 - 1) << 1$).

Then, for $k<<1$ (curvature of the axis):

$$|k_{CS}| \approx (\upsilon k x_0)k = \alpha k << k$$

the factor α varies from *0* at $x_0=0$ (the member symmetry plane, without warping) to $\upsilon kl/2 = \upsilon l/2R$<<1 at end $x_0=l/2$ ($R=1/k$, curvature radius of the axis). Thus, the curvature of CS is much smaller than the curvature of the axis (itself very small), and it justifies the simplifying hypothesis of flat CS. Note that a CS curvature of the same order of magnitude as that of the axis requires big displacements, i.e. for $R = l/2$ (Figure 5) where we find that $\alpha = \upsilon$ at the bar end, and CS curvature is υ times that of the axis; but big displacements require other formulation, deformed axis is no longer a parabola and neglected terms in the denominator of (10) are no longer negligible.

Figure 5. Curvature of CS for big displacements. Source: Self-Elaboration.



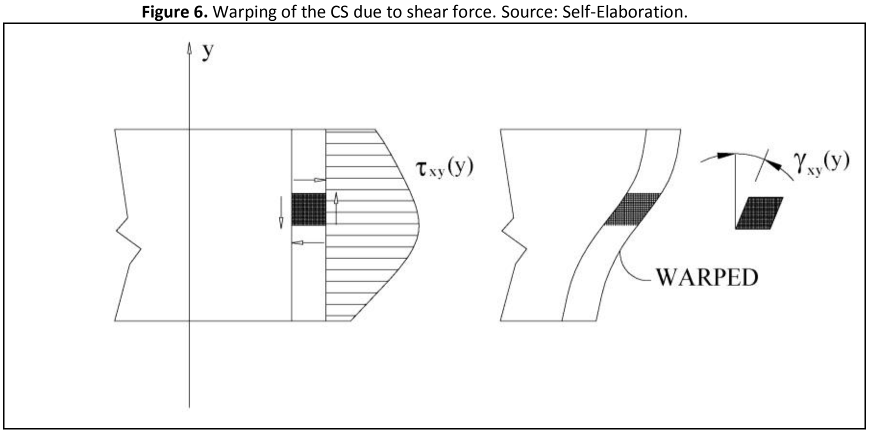
In the case of pure bending, the fact is that CS warping does not affect the precision of some simplified calculations made as if it were flat: only normal stresses on the whole CS and displacements of axis material points (*y=z=0*) fully coincide with exact values.

Shear bending

Discussion in this case is shorter than for the two precedent cases, and does not pose any problem.

Well known elastic solutions available in cited literature for elemental cases of bending with shear (in *y* direction, for example) include, in addition to σ_x, variable τ_{xy} stress component along transverse direction (Figure 6). It produces variable strains γ_{xy} that are directly responsible for warping: the cross section will never remain flat in presence of shear force.

Figure 6. Warping of the CS due to shear force. Source: Self-Elaboration.



In this case, the Navier-Bernoulli hypothesis always leads to linear normal stresses σ_x, but exact solution is nonlinear for some well known cases; transverse displacements of the axis material points calculated applying this hypothesis are always smaller than the exact values, due to shear strain. Differences in exact values of stresses and displacements have the same order of magnitude as depth/length ratio squared, being negligible for slender bars as it can be seen in referred treatises.

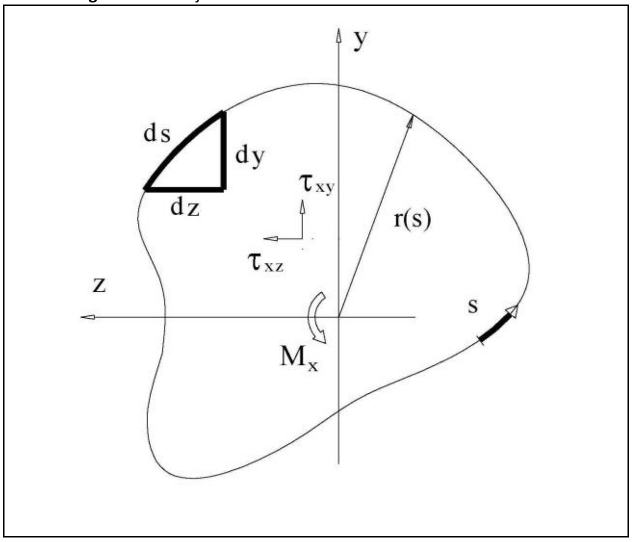
Torque

In spite of Navier-Bernoulli simplifying hypothesis not being applicable to torsion problem in prismatic members, discussion in this case is also shorter than for the axial and bending cases, and again do not pose any problem.

In general, torque leads to warping of CS, its magnitude depending on its shape.

For nonuniform torsion (with variable warping) it is evident that CS cannot remain flat. For the case of uniform torsion (with free warping) elastic available solution in cited elasticity references gives (Figure 7).

Figure 7. CS subjected to uniform torsion. Source: Self-Elaboration.



$$\boldsymbol{T} = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ & 0 & 0 \\ SIM & & 0 \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} 0 & \tau_{xy}/2G & \tau_{xz}/2G \\ & 0 & 0 \\ SIM & & 0 \end{bmatrix}$$

$$u = \theta \Psi(y,z)$$
$$v = -\theta z x$$
$$w = \theta y x$$

With

$$\theta = M_x / GI_t$$

$\Psi(y,z)$ = warping function

I_t being the torsion modulus, G the transverse shear modulus, and

$$\Delta\Psi = \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = 0 \quad \text{in the entire CS} \tag{11.a}$$

$$(\frac{\partial\Psi}{\partial y} - z)\frac{dz}{ds} - (\frac{\partial\Psi}{\partial z} + y)\frac{dy}{ds} = 0 \quad \text{on the CS boundary} \tag{11.b}$$

Flat deformed CS requires $\Psi \equiv 0$ that satisfies directly (11.a), but (11.b) requires:

$$0 = z\frac{dz}{ds} + y\frac{dy}{ds} = \frac{1}{2}\frac{d(y^2+z^2)}{ds} = \frac{1}{2}\frac{d(r^2)}{ds} \Rightarrow r(s) \text{ constant at CS boundary}$$

and it can only apply to cases in which the CS is circular. Any other shape when subjected to torsion shows warping displacements.

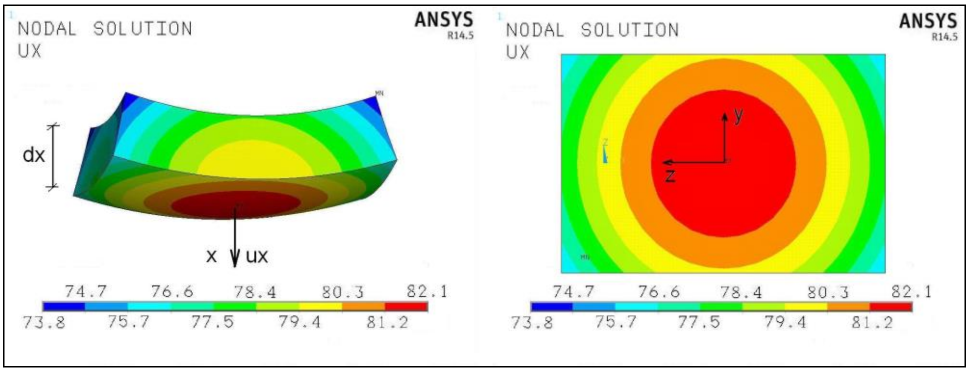
The mathematical model proposed in *Strength of Materials* makes use of varied simplifying hypotheses to calculate either constant or variable warping along the axis.

Numerical illustration

Since torsion and shear bending pose no problem, we illustrate two examples for axial load and pure bending of a rectangular CS bar by mean of elastic linear solutions we obtained from finite element models with ANSYS (© ANSYS, 2014), enlarging the scale to better appreciate warping. The examples confirm the existence of CS warping demonstrated in our equations. Models were built with hexahedral three-dimensional elements of 8 nodes.

Figure 8 shows the paraboloid (a) corresponding to a linear axial force along the axis (its own weight acting, $\rho \neq 0$); CS frontal print (b) maps the axial displacements and verifies that it is a revolution paraboloid as indicated by (5).

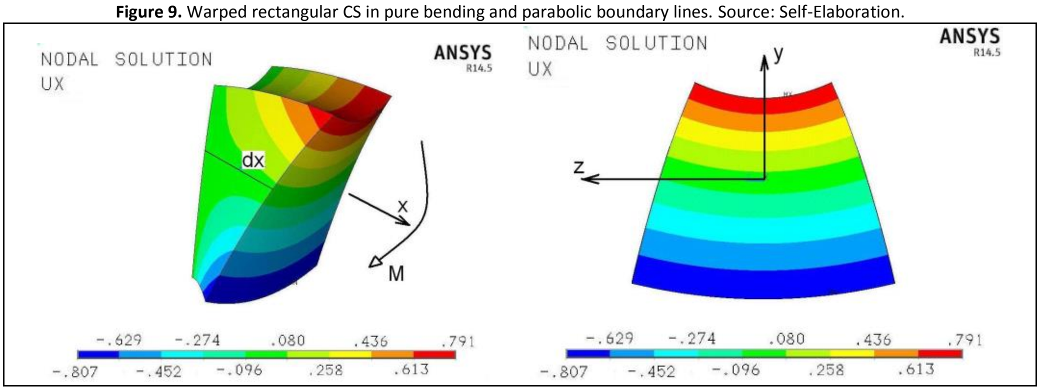
Figure 8. Revolution paraboloid for deformed CS under linear axial force. Source: Self-Elaboration.



a) b)

Figure 9 shows a perspective (a) of the warped CS for pure bending; CS frontal view (b) maps the axial displacements that generate parabolic lines for straight lines parallel to principal inertia CS axes in the undeformed geometry.

Figure 9. Warped rectangular CS in pure bending and parabolic boundary lines. Source: Self-Elaboration.



a) b)

Conclusion

From our precedent analysis, the following conclusions can be extracted for prismatic structural elastic members in the mathematical model corresponding to small strains, displacements and rotations.

a) Warping of cross section no doubt occurs in the presence of shear or nonuniform torsion.
b) Even in the elemental case of linear axial force, CS does not remain flat after deformation, warping to a revolution paraboloid (for any CS shape).
c) In spite of the conclusions of many specialised treatises and the belief of many professionals, even in the elemental case of pure bending, CS does not remain flat after deformation; warping occurring to a 4^{th} order polynomial surface containing two parabolas families (for any CS shape), as proved in the paper.

d) The question whether warping occurs or not derives from two motives. First, some *Strength of Materials* symmetry considerations usually made do not specify the possibility of free warping at bar ends, concluding a non-existent symmetry for each half of the bar that leads to a flat CS. Second, some *Elasticity* treatises hastily interpret deformed surface for CS based directly on the axial displacement its points undergo as a function of undeformed coordinates *(x, y, z)*. Correct interpretation requires the analysis of relationship between the deformed coordinates *(x', y', z')* as it is explained in the paper.
e) In this mathematical model, cross section remains exactly flat only in the cases of constant axial force (for any CS shape) and uniform torsion for circular CS only.
f) Although CS does no remain exactly flat in some cases, its curvature is much smaller than the axis curvature (itself very small in bending), and it justifies the simplifying hypothesis of flat CS.

References

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Argüelles, A., & Viña, I. (2004). *Elasticidad y Resistencia de Materiales*. (Bellisco, Ed.) (1st ed.). Madrid.

Feodosiev, V. I. (1980). *Strength of Materials*. (MIR, Ed.) (2nd ed.). Moscow.

Samartin, A. (1990). *Curso de Elasticidad*. (Bellisco, Ed.) (1st ed.). Madrid.

Timoshenko, S. (1940). *Strength of Materials (part I)*. (V. Nostrand, Ed.) (2nd ed.). New York.

Timoshenko, S., & Goodier, J. N. (1951). *Theory of Elasticity*. (McGraw-Hill, Ed.) (2nd ed.). New York.

Torroja, E. (1967). *Lecciones elementales de Elasticidad con aplicación a la técnica de la construcción*. (Dossat, Ed.) (4th ed.). Madrid.